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Algebraic description of a two-dimensional system of charged particles in an external magnetic field and periodic potential

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Abstract. Properties of the magnetic translation operators for a charged particle moving in a crystalline potential and a uniform magnetic field show that it is necessary to consider *all* inequivalent irreducible projective representations of the crystal lattice translation group. These considerations lead to the concept of magnetic cells and indicate the periodicity of physical properties with respect to the charge. It is also proven that a direct product of such representations describes a system of two (many, in general) particles. Therefore, they can be applied in a description of interacting electrons in a magnetic field, for example in the fractional quantum Hall effect.

1. Introduction

The magnetic translation operators

$$T(\boldsymbol{R}) = \exp\left[-\frac{\mathrm{i}}{\hbar}\boldsymbol{R}\cdot\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)\right]$$

introduced by Brown (1964), to describe the movement of a Bloch electron in an external magnetic field, form in fact a projective (ray) representation of the translation group with a factor system (Brown 1964, Zak 1964a, b)

$$T(\boldsymbol{R})T(\boldsymbol{R}')[T(\boldsymbol{R}+\boldsymbol{R}')]^{-1} = m(\boldsymbol{R},\boldsymbol{R}') = \exp\left[-\frac{1}{2}\frac{\mathrm{i}e}{c\hbar}(\boldsymbol{R}\times\boldsymbol{R}')\cdot\boldsymbol{H}\right]$$

where $\boldsymbol{H} = \boldsymbol{\nabla}\times\boldsymbol{A}$. This is only one of many applications of projective representations, firstly investigated by Schur (1904, 1907, 1911), in quantum physics. However, its clarity and importance led Backhouse and Bradley to start their series of articles on projective representations with this example (Backhouse and Bradley 1970, Backhouse 1970, 1971, Backhouse and Bradley 1972). Another important application is illustrated by the construction of space groups (Altmann 1977); however, in this case one considers projective representations of the point group (see also Bradley and Cracknell 1972).

The other—equivalent—description of Bloch electrons in a magnetic field was proposed by Zak (1964a, b) and applied, e.g., by Divakaran and Rajagopal (1995) and the author (Florek 1994, 1996a, b). This approach consists in the introduction of a covering group and investigations of its ordinary, i.e. vector, representations (see also Altmann 1977, 1986). The covering group contains pairs $(\alpha, \boldsymbol{R})$, $\alpha\in\mathrm{U}(1)$, and its vector representation can be written as a product $\Gamma(\alpha)T(\boldsymbol{R})$, where Γ is a representation of U(1) and T is a projective representation of the

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translation group (Zak 1964a, Altmann 1977, Florek 1994). Zak rejected representations with $\Gamma(\alpha) \neq \alpha$ as 'non-physical' (Zak 1964b). However, if T' is a projective representation with a factor system $\Gamma(m(\boldsymbol{R}, \boldsymbol{R}'))$, then the product $\Gamma T'$ is a vector representation of the covering group and there are no rules that are contravened by considering this case. The first attempt to consider *all* representations was performed within Zak's approach by the author (Florek 1997a); in that work, the physical consequences of taking into account all cases were indicated.

This paper is based on Brown's approach; i.e. projective representations of the translation group are considered. It is shown that all projective representations are necessary in a description of the movement of a particle with the charge qe, where q is an integer, in a magnetic field and a crystalline potential. Moreover, applying the results of earlier articles (Florek 1997b, Florek and Wałcerz 1998), this is done for any vector potential $\boldsymbol{A}$ (strictly speaking, for $\boldsymbol{A}$ a linear function of the coordinates; however, by appropriate gauge transformation each vector potential can be written in such a form for a constant, uniform magnetic field). This removes the restriction imposed by Brown (1964) and Zak (1964a, b) on $\boldsymbol{A}$ of being a fully antisymmetric function of the coordinates (i.e. $\partial A_l/\partial x_k + \partial A_k/\partial x_l = 0$ for each pair $k, l = 1, 2, 3$). Moreover, the proposed approach yields in a natural way the concept of magnetic cells (Zak 1964a, b) and proves the periodicity of physical properties with respect to the charge, in addition to the periodicity with respect to the magnitude of the magnetic field proven by Azbel (1963). Since projective representations correspond to energy levels of one-particle states, their direct products must describe two-particle states (or many-particle states in a more general case). A system of two particles with the charges qe and $q'e$ has the total charge $(q + q')e$ and, therefore, should correspond to a projective representation with a factor system determined by this charge. It follows from the previous discussion that in a many-body problem one has to consider all representations, including those considered by Zak as 'non-physical'.

2. Periodicity with respect to charge

The Hamiltonian describing the motion of a charged particle in a periodic potential $V(\boldsymbol{r})$ and an external magnetic field $\boldsymbol{H} = \boldsymbol{\nabla} \times \boldsymbol{A}$ is given as

$$\mathcal{H} = \frac{1}{2m}\left(\boldsymbol{p} - \frac{qe}{c}\boldsymbol{A}\right)^2 + V(\boldsymbol{r})$$

where m denotes the effective particle mass, $\boldsymbol{p}$ its kinetic momentum, and qe, with $q \in \mathbb{Z}$ and $e > 0$, its charge. If the vector potential $\boldsymbol{A}$ is a linear function of the coordinates, i.e.

$$A_\alpha = \sum_\beta a_{\alpha\beta}\beta \qquad \alpha, \beta = x, y, z$$

then the magnetic translation operators can be written as (Florek 1997b, Florek and Wałcerz 1998)

$$T(\boldsymbol{R}) = \exp\left[-\frac{\mathrm{i}}{\hbar}\boldsymbol{R}\cdot\left(\boldsymbol{p} - \frac{qe}{c}\boldsymbol{A}'\right)\right]$$

where $\boldsymbol{A}'$ is a vector potential associated with $\boldsymbol{A}$, defined as

$$A_\alpha = \sum_\beta a_{\beta\alpha}\beta.$$

It is well known (Brown 1964, Zak 1964a, b) that the periodic boundary conditions allow us to consider a two-dimensional crystal lattice (in the xy-plane, say) and $\boldsymbol{H} = [0, 0, H]$ perpendicular to it. Hence, any lattice vector can be considered as two-dimensional:

$$\boldsymbol{R} = n_1\boldsymbol{a}_1 + n_2\boldsymbol{a}_2.$$

The magnetically periodic boundary conditions (Brown 1964) yield quantization of a magnetic flux:

$$\boldsymbol{H} \cdot (\boldsymbol{a}_1 \times \boldsymbol{a}_2) = \frac{2\pi}{N} \frac{\hbar c}{e} \frac{L}{q}$$

where the integer L is mutually prime with the crystal period N. Replacing the left-hand side by the flux per the unit cell

$$\phi = (e/hc)\boldsymbol{H} \cdot (\boldsymbol{a}_1 \times \boldsymbol{a}_2)$$

one obtains

$$N\phi = \frac{L}{q}. \tag{1}$$

The factor system $m(\boldsymbol{R}, \boldsymbol{R}')$ depends on $\boldsymbol{A}$: for example, the antisymmetric gauge $\frac{1}{2}(\boldsymbol{H} \times \boldsymbol{r})$ gives (Brown 1964, Zak 1964a, b)

$$m(\boldsymbol{R}, \boldsymbol{R}') = \omega_N^{(1/2)L(n_2 n_1' - n_1 n_2')} = \omega_{2N}^{L(n_2 n_1' - n_1 n_2')} \tag{2}$$

whereas for the Landau gauge $\boldsymbol{A} = [0, Hx, 0]$ (and $\boldsymbol{A}' = [-Hy, 0, 0]$)

$$m_N^{(L)}(\boldsymbol{R}, \boldsymbol{R}') = \omega_N^{L n_2 n_1'}. \tag{3}$$

In both formulae, $\omega_N = \exp(2\pi\mathrm{i}/N)$. However, the group-theoretical commutator is gauge independent, and for any linear gauge we have (Florek and Wałcerz 1998)

$$T(\boldsymbol{R})T(\boldsymbol{R}')T^{-1}(\boldsymbol{R})T^{-1}(\boldsymbol{R}') = \omega_N^{-L(n_1 n_2' - n_2 n_1')}. \tag{4}$$

The matrices of the irreducible projective representation corresponding to the factor system given as (3) can be chosen as

$$D_{ij}^{NL}(\boldsymbol{R}) = \delta_{i,j-n_2}\omega_N^{L n_1 i} \qquad i, j = 0, 1, \ldots, N-1. \tag{5}$$

It should be underlined that such a projective representation is normalized (cf. Altmann 1977, 1986, Florek and Wałcerz 1998), in contrast to those corresponding to the factor system (2) and considered by Brown (1964).

If $\gcd(L, N) = \nu > 1$, the representations (5) are reducible and the corresponding factor system is

$$m_n^{(l)}(\boldsymbol{R}, \boldsymbol{R}') = \omega_n^{l n_2 n_1'} \tag{6}$$

where $l = L/\nu$, $n = N/\nu$, and $\gcd(l, n) = 1$. Irreducible projective representations with such factors have to be n-dimensional, which directly leads to the concept of *magnetic cells*: one obtains $D^{NL}(n\boldsymbol{R}) = \mathbf{1}$, so the *magnetic period* is equal to n, though the *crystal period* is still N. Therefore, the $N \times N$ lattice can be viewed as a $\nu \times \nu$ lattice, with the translation group $T_\nu = \mathbb{Z}_\nu^2$, of $n \times n$ magnetic cells. Let (ξ_1, ξ_2) label magnetic cells, whereas (η_1, η_2) is the position within a magnetic cell, i.e. $n_i = \eta_i + \xi_i n$. Then the matrices

$$D_{ij}^{nl,\boldsymbol{k}}(\boldsymbol{R}) = D_{ij}^{nl}(\eta_1, \eta_2) D^{\boldsymbol{k}}(\xi_1, \xi_2) = \delta_{i,j-\eta_2}\omega_n^{l\eta_1 i} D^{\boldsymbol{k}}(\xi_1, \xi_2) \tag{7}$$

form an irreducible projective representation of $\mathbb{Z}_N^2$ with the factor system (6), where

$$D^{\boldsymbol{k}}(\xi_1, \xi_2) = \exp[-2\pi\mathrm{i}(k_1\xi_1 + k_2\xi_2)/\nu] = \omega_\nu^{-(k_1\xi_1 + k_2\xi_2)} \tag{8}$$

is an irreducible representation of T_ν (Backhouse 1970). The character of the representation given by (7) is

$$\chi_{n,l;\boldsymbol{k}}(\boldsymbol{R}) = \delta_{\eta_1,0}\delta_{\eta_2,0} n \omega_\nu^{-(k_1\xi_1 + k_2\xi_2)}.$$

For given n and l (i.e. for a given factor system), we obtain all ν^2 inequivalent irreducible projective representations labelled by $\boldsymbol{k}$ (Altmann 1977, 1986), and all of them are normalized.

To determine a relation between the charge q of a particle and the irreducible projective representation $D^{nl,\boldsymbol{k}}$, let us fix the magnetic flux ϕ and the crystal period N. Then the condition (1) gives that $L = N\phi q$; i.e. $L \propto q$. However, this is not a one-to-one relation, since L is limited to the range $0, 1, \ldots, N-1$ with no condition imposed on $q \in \mathbb{Z}$. The representation (5), its factor system (3), and the commutator (4) are determined by ω_N^L, so all of them are periodic functions of $L \propto q$, and, therefore, periodic functions with respect to the charge of a moving particle. We see, in particular, that for $q = zN$, $z \in \mathbb{Z}$, vector representations with trivial factor systems (and trivial commutators) are obtained. This means that for a given crystal period N and constant magnetic field, a particle with the charge zNe behaves as a non-charged one. It is also easy to see that for some q we can obtain $L = l\nu$, where $\nu = \gcd(N, L)$, and in this case the irreducible representations $D^{nl,\boldsymbol{k}}$ have to be used. Since ν is a co-divisor of n, then assuming $\phi = 1/N$ we obtain

$$q = N\frac{l}{n} \tag{9}$$

which relates the pair (n, l) (the label of the irreducible representation) and the charge q of a particle. It has to be underlined that this relation has been derived for a fixed ϕ and does not depend on the irreducible representations $D^{\boldsymbol{k}}$ of T_ν given by (8).

3. Multi-particle states

It can be shown (see, for example, Altmann 1986) that a product of two projective representations D' and D'' of a given group G with factor systems m' and m'', respectively, is another projective representation with a factor system $m(g, g') = m'(g, g')m''(g, g')$, which, in general, is different from factor systems m' and m''. Let D be a product of two irreducible projective representations $D^{nl,\boldsymbol{k}}$ and $D^{n'l',\boldsymbol{k}'}$. Then their product has a factor system

$$m(\boldsymbol{R}, \boldsymbol{R}') = \omega_N^{Ln_2n_1'} \qquad \text{with } L = l\nu + l'\nu' \tag{10}$$

so it corresponds to the representation $D^{NL,\boldsymbol{K}}$ ($\boldsymbol{K}$ has not been determined, but it depends on the irreducibility of the representation obtained). The character of this representation is

$$\chi(\boldsymbol{R}) = \delta_{\eta_1,0}\delta_{\eta_2,0}\delta_{\eta_1',0}\delta_{\eta_2',0}nn'\omega_N^{-n(k_1\xi_1+k_2\xi_2)-n'(k_1'\xi_1'+k_2'\xi_2')}$$

so it is non-zero only for $n_i = x_i m$, where $m = nn'/\gamma$, $\gamma = \gcd(n, n')$, $0 \leqslant x_i < \mu = N/m = \gcd(\nu, \nu')$. Substituting m and μ into the above formula, one obtains

$$\chi(\boldsymbol{R}) = \delta_{\eta_1,0}\delta_{\eta_2,0}m\gamma\omega_\mu^{-(k_1+k_1')x_1-(k_2+k_2')x_2} \qquad (\mathrm{mod}\ m). \tag{11}$$

Since $\nu/\mu = n'/\gamma$, then L in (10) can be written as

$$L = \mu\left(\frac{l\nu}{\mu} + \frac{l'\nu'}{\mu}\right) = \mu\left(\frac{ln'}{\gamma} + \frac{l'n}{\gamma}\right) = \mu\lambda. \tag{12}$$

It seems that this determines a factor system $m_m^{(\lambda)}$. However, we cannot exclude the case in which $\gcd(\lambda, m) = \ell > 1$. Therefore, the product considered has to be decomposed into irreducible representations with a factor system $m_M^{(\Lambda)}$, where $\Lambda = \lambda/\ell$ and $M = m/\ell$. The scalar product of the appropriate characters gives us a multiplicity of $D^{M\Lambda,\boldsymbol{K}}$ in the product considered, as follows:

$$f(D^{M,\Lambda;\boldsymbol{K}}, D^{nl,\boldsymbol{k}} \otimes D^{n'l',\boldsymbol{k}'}) = \frac{\gamma}{\ell}\delta_{K_1,k_1+k_1'}\delta_{K_2,k_2+k_2'}. \tag{13}$$

There are ℓ^2 such representations with $K_i = (k_i + k_i') \bmod \mu$.

The most interesting is the case in which $n = n'$ and $l = l'$, since n and l are determined by the magnetic flux, the charge, and the crystal period N; hence such a case can be interpreted as a system of two identical particles moving in the same lattice and the same magnetic field (Florek 1997a). The resultant representation is n^2-dimensional and its character is equal to

$$\chi(\boldsymbol{R}) = \delta_{n_1,x_1 n}\delta_{n_2,x_2 n} n^2 \omega_\nu^{-(k_1+k_1')x_1-(k_2+k_2')x_2}$$

with $0 \leqslant k_i, k_i', x_i < \nu$. The factor system is given by (10) as

$$m(\boldsymbol{R}, \boldsymbol{R}') = \omega_n^{2ln_2n_1'} = \omega_n^{\lambda n_2 n_1'}$$

so we have to check the $\gcd(\lambda, n)$. At this point, the cases of odd and even n have to be considered separately. In the first, case $\ell = \gcd(n, 2l) = 1$, and the representation obtained decomposes into n copies of the representation $D^{n2l,\boldsymbol{K}}$ with $K_i = (k_i + k_i') \bmod \nu$. In the second case, however, $\ell = 2$ and $M = \frac{1}{2}n$, so the product considered decomposes into the representations $D^{(1/2)nl,\boldsymbol{K}}$: there are four representations with $K_i = (k_i + k_i') \bmod \nu$ and each of them appears $\frac{1}{2}n$ times. In both cases we have

$$\frac{2l}{n} = \frac{l}{(n/2)} = 2\frac{l}{n}$$

so the new representations correspond to a system with the charge $2q$; see (9). However, an even n in the second case yields the change of magnetic periodicity from n to $\frac{1}{2}n$ and four times as many magnetic cells. In a similar way, the coupling of d representations $D^{n1,\boldsymbol{k}^{(j)}}$, $j = 1, 2, \ldots, d$ with $n = dM$, changes the magnetic period from n to M (and yields d^2 times as many magnetic cells)—however, not by modification of the magnetic field, but by multiplication of the charge by d.

The irreducible representations (7) are written as a product of a one-dimensional irreducible representation $D^{\boldsymbol{k}}$ of T_ν, equation (8), and a projective one of T_n. This means that products of such representations can also be separated into a part describing addition of the quasi-momenta $\boldsymbol{k}, \boldsymbol{k}'$ with the second part corresponding to the addition of co-divisors ν and ν' or, more precisely, $l\nu + l\nu'$; see (10). However, the last addition can change the magnetic periodicity, determined by M and Λ in (12) and (13), in a way depending on the arithmetic structure of N, n, n', l, and l'. In the above example, the label M (the size of the magnetic cells) of the resultant representation was equal to or smaller than $n = n'$. One can easily obtain that for $N = 12$

$$D^{3,1;[1,0]} \otimes D^{6,1;[1,0]} = \bigoplus_{K_1,K_2=0,2,4} D^{2,1;[K_1,K_2]}.$$

In this case one particle may have charge $4e$ and the second $2e$, so the two-particle system has the charge $6e$. We must say 'may have' since condition (1) involves both the magnetic flux ϕ and the charge q. The chosen values of the charges correspond to the fixed $\phi = 1/N$. Therefore, the charge of the first particle yields 3×3 magnetic cells, and the second one 6×6, whereas the two-particle system demands 2×2 magnetic cells. On the other hand we have ($N = 12$, as above)

$$D^{3,1;[1,0]} \otimes D^{4,1;[1,0]} = D^{12,7;[0,0]}$$

so $M > n, n'$ and there is only one magnetic cell. Therefore, the addition of quasi-momenta $\boldsymbol{k}, \boldsymbol{k}'$ has to be modified to reflect all possible changes of the magnetic periodicity.

4. Final remarks and conclusions

The projective representations used by Brown (1964) and in this paper can be replaced in an equivalent approach by using vector representations of central extensions (Zak 1964a, b, Florek 1994, 1996a, b). Zak assumed that a factor ω has to be represented by itself, and rejected representations in which ω is represented by ω^r. However, as long as r is mutually prime with N, such a change is an isomorphism of (inequivalent) central extensions (Altmann 1977, Florek 1994). Within the approach presented here, this fact is realized by the freedom that one has in choosing the relation between the charge q and the index l, given by (9). For $q = 1$, we can take not only $L = 1$ but also any r mutually prime with n. All important properties, e.g. the addition of charges and charge periodicity, are unaffected: since $\gcd(r, n) = 1$, then

$$\{r, 2r, \ldots, Nr\} = \{1, 2, \ldots, n\}$$

but elements of the first set are obtained in a different order (zr is calculated $\bmod n$). In physical terms, this means that if we observe only magnetic or charge periodicity, we cannot distinguish $H_1 = 2\pi\hbar c/Ne$ from $H_r = rH$ if $\gcd(r, N) = 1$; see (1) and (9). In fact, it should be said that condition (1) is not imposed on H or q but on their product qH, and has to be written as

$$qH = \frac{2\pi}{N}\frac{\hbar c}{e}L \qquad \text{or} \qquad q\phi = \frac{L}{N}. \tag{14}$$

This means that a particle with the charge $2e$ can be described by the same representation D^{NL} as a particle with the charge e if the magnetic field is halved. On the other hand, very strong magnetic fields may lead to observation of a fractional charge, if the product qH has to satisfy (14).

The introduction of projective representations in this paper has been based on the magnetic translation operators determined by Brown (1964), and the notion of Bloch electrons in an external magnetic field was used throughout this work. Hence, the concept of magnetic cells has appeared in a natural way. However, these representations can be applied to any problem in quantum mechanics in which a symmetry group G appears and phase factors play an important role. For example, Divakaran and Rajagopal (1991) used them in the theory of superconducting layered materials (they also included many general remarks in their work). If we assume that projective representations correspond to energy levels (and so representation vectors correspond to states) of a one-particle system, then products of two (or more) representations have to correspond to two-particle (or many-particle, in a general case) systems. Not straying far from the physical problems discussed above, we can look at a two-dimensional electron gas in an external magnetic field. The fractional quantum Hall effect (Tsui *et al* 1982, Das Sarma and Pinczuk 1997) is still a subject to which much effort is being devoted by theorists and experimentalists, but it has been accepted that Coulomb interactions play a very important role in the explanation of the observed features (Shankar and Murthy 1997, Heinonen 1998). Therefore, it seems possible to apply the results presented above to such problems.

It should be underlined that products of *projective* representations are well known in mathematics (Backhouse and Bradley 1972, Altmann 1986). On the other hand, products of *vector* representations are commonly used in quantum physics to describe multi-particle states. It is shown in this paper that *products of projective* representations also have to be applied in many-body problems.

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